Math 210A Lecture 7 Notes

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1 Representable Functors and Free Groups

1.1 Representable functors

Definition 1.1. A contravariant functor $F : \mathcal{C} \to \text{Set}$ is **representable** if there is a natural isomorphism $h^B \to F$ for some $B \in \mathcal{C}$, where $h^B = \text{Hom}_{\mathcal{C}}(\cdot, B)$.

Example 1.1. Let $P : \text{Set} \to \text{Set}$ be the morphism such that $P(S) = \mathcal{P}(S)$, the power set of S, and $P(f : S \to T)(V) = f^{-1}(V)$ for $V \subseteq T$. P is representable by $\{0, 1\}$; $P(S) \xrightarrow{\sim} \text{Maps}(S, \{0, 1\})$, which sends $U \mapsto \mathbb{1}_U$, the indicator function of U.

$$\begin{tikzcd}
P(T) \arrow[r, "\sim"] \arrow[d, "P(f)"] & \text{Maps}(T, \{0,1\}) \arrow[d, "h^{\{0,1\}}(f)"] \\
P(S) \arrow[r, "\sim"] & \text{Maps}(S, \{0,1\})
\end{tikzcd}$$

Lemma 1.1. *A representable functor is represented by a unique object up to (unique) isomorphism. That is, if B, C represent $F : \mathcal{C} \to \text{Set}$, then there exists a unique isomorphism $f : B \to C$ such that*

$$\begin{tikzcd}
\text{Hom}_{\mathcal{C}}(A, B) \arrow[r, "\sim"] \arrow[d, "h_A(f)"] & F(A) \arrow[d, "\text{id}_A"] \\
\text{Hom}_{\mathcal{C}}(A, C) \arrow[r, "\sim"] & F(A)
\end{tikzcd}$$

Proof. There exist natural isomorphisms $\xi : h^B \to F$, $\xi' : h^C \to F$. Then $(\xi')^{-1} \circ \xi$ is a natural isomorphimsm $h^B \to h^C$. Yoneda's lemma gives a unique $f : B \to C$ such that $h^{\mathcal{C}}(f) = (\xi')^{-1} \circ \xi$ because $h^{\mathcal{C}}(f)_A = h_A(f)$. $\square$

Remark 1.1. A covariant functor $F : \mathcal{C} \to \text{Set}$ is representable if there exists a natural isomorphism $F \to h_A$ for some $A \in \mathcal{C}$.

Example 1.2. Let $\Phi : \text{Grp} \to \text{Set}$ be the forgetful functor. To represent Φ, we want a bijection $\Phi(G) = G \xrightarrow{\sim} \text{Hom}_{\text{Grp}}(\mathbb{Z}, G)$; send $g \mapsto (n \mapsto g^n)$. This image homomorphism is completely determined by whatever 1 gets sent to, which is g. So this is a bijection. So Φ is represented by $\mathbb{Z}$.

1.2 Free groups

Definition 1.2. A group F is **free** on a subset $X \subseteq F$ if for any function $f : X \to G$, where G is a group, there exists a unique homomorphism $\phi_f : F \to G$ such that $\phi_f(x) = f(x)$ for all $x \in X$.

Example 1.3. Let $\Phi : \text{Grp} \to \text{Set}$ be the forgetful functor. If $f \in \text{Hom}_{\text{Set}}(X, \Phi(G)) = \text{Maps}(X, G)$, we want $\phi_f \in \text{Hom}_{\text{Grp}}(F_X, G)$, where F_X is the free group on X. We want a bijection $\text{Hom}_{\text{Grp}}(F_X, G) \xrightarrow{\sim} \text{Hom}_{\text{Set}}(X, \Phi(G))$. Send $\phi \mapsto \phi|_X$. If $f : G \to H$ is a homomorphism,

$$\begin{tikzcd}
\text{Hom}_{\text{Grp}}(F_X, C) \arrow[d, "\phi_f \mapsto \varphi \circ \phi_f"] & \text{Maps}(X, G) \arrow[l, "\sim"'] \arrow[d, "f \mapsto \phi \circ f"] \\
\text{Hom}_{\text{Grp}}(F_X, H) & \text{Maps}(X, H) \arrow[l, "\sim"']
\end{tikzcd}$$

If F_X exists for all X, then $F : \text{Set} \to \text{Grp}$ with $F(X) = F_X$ and $F(\varphi)$ the unqiue morphism is left adjoint to Φ. Why is this morphism unique? $\varphi : X \to Y$ induces a map $h : X \to F_Y$. There exists a unique map $\phi_h : F_X \to F_Y$ by the universal property.

Definition 1.3. Let $\Phi : \mathcal{C} \to \text{Set}$ be a faithful functor and X a set. A **free object** F_X on X in $\mathcal{C}$ is a function $\iota : X \to \Phi(F_X)$ such that $\text{Hom}_{\mathcal{C}}(F_X, B) \xrightarrow{\sim} \text{Maps}(X, \Phi(B))$ via $\alpha \mapsto \Phi(\alpha) \circ \iota$ is a bijection for all $B \in \mathcal{C}$.

Example 1.4. The forgetful functor $\Phi : \text{Top} \to \text{Set}$ takes a topological space and returns the underlying set, forgetting the topology. Let's find a left adjoint. If X is a set, we can map it to a topological space $F_X = X$ with the discrete topology. Then $\text{Hom}_{\text{Top}}(X, B) = \text{Maps}(X, B)$.

Example 1.5. Let $\Phi : \text{Ab} \to \text{Set}$ be the forgetful functor. Let $\iota : X \to \bigoplus_{x \in X} \mathbb{Z}$ send $x \mapsto 1 \cdot x$. We want a bijection $X \mapsto \bigoplus_{x \in X} \mathbb{Z}$. $\text{Hom}_{\text{Ab}}(\bigoplus_{x \in X} \mathbb{Z}, B) \to \text{Maps}(X, B)$. For the backwards direction, send $f \mapsto \phi_f(\sum_x a_x x) = \sum_x a_x f(x)$. In the forward direction, we have $\phi \mapsto (x \mapsto \phi(1 \cdot x))$. $\bigoplus_{x \in X} \mathbb{Z}$ is called the **free abelian group** on X.

How do the free group X and the free abelian group $\bigoplus_{x \in X} \mathbb{Z}$ compare? There is a surjective homomorphism $F_X \to \bigoplus_{x \in X} \mathbb{Z}$ sending $x \mapsto 1 \cdot x$. This is because we have the bijection $\text{Hom}_{\text{Grp}}(F_X, \bigoplus_{x \in X} \mathbb{Z}) \xrightarrow{\sim} \text{Maps}(X, \bigoplus_{x \in X} \mathbb{Z})$. We can't go the other way because a free group is not necessarily abelian.